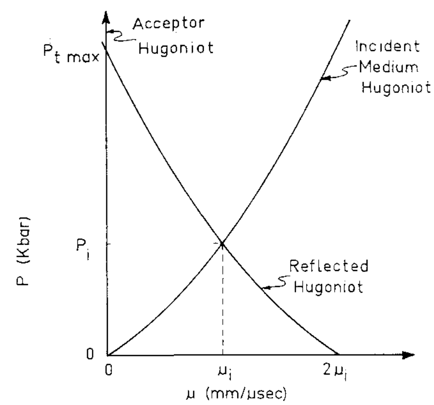

Fig 1 Hugoniot "reflection" method

mentioned approach gives $(P_t/P_i)_{max} = 8$ It should also be pointed out that the result also depends on the inherent accuracy of the Hugoniot "reflection" method itself

In order to illustrate some of the foregoing remarks, (P_t/P_i) was computed for shock transmission from Plexiglas[9] to iron[10] and Plexiglas to platinum[10] using both the "impedance-mismatch" approximation [Eq (1)] and the Hugoniot "reflection" method The results are shown in columns 1–5 of Table 1 It is seen that Eq (1) gives quite low values in this region and that the ratio can, as indicated, be greater than two at finite pressures in real materials Columns 6 and 7 in Table 1 show the values of $(P_{t\,max}/P_i)$ as computed from Eq (14) compared to the maximum value from Eq (1) Clearly, increases in P_i give increases in $(P_{t\,max}/P_i)$, approaching the limiting value of 4 computed in Eq (18)

Table 1 Comparison of values of (P_t/P_i) for shock transmission from Plexiglas to iron and platinum computed by the "impedance-mismatch" approximation and the Hugoniot "reflection" method

Plexiglas $\rho_i = 1\ 18$ g/cm³ P_i, kbar	Iron $\rho_t = 7\ 84$ g/cm³ $\left(\frac{P_t}{P_i}\right)^a$	$\left(\frac{P_t}{P_i}\right)^b$	Platinum $\rho_t = 21\ 37$ g/cm³ $\left(\frac{P_t}{P_i}\right)^a$	$\left(\frac{P_t}{P_i}\right)^b$	Ordinate $\rho_t \rightarrow \infty$ $\left(\frac{P_t}{P_i}\right)^a$	$\left(\frac{P_{t\max}}{P_i}\right)^c$
102 7	1 70	2 14	1 86	2 61	2 00	3 13
162 4	1 66	2 12	1 83	2 56	2 00	3 25
235 0	1 65	2 11	1 83	2 60	2 00	3 35

[a] Computed from impedance mismatch approximation Eq (1)
[b] Computed by Hugoniot "reflection method
[c] Computed from Eq (14)

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Stability of the Hodograph Equations in One-Dimensional Reacting Gas Flow

JAMES L THRONE*
University of Delaware, Newark, Del

IT can be shown that the standard equations describing flow of reacting gases in nozzles have singularities at the Mach number equal to unity Various methods have been devised for passing through this point, including extrapolation[1] and lowering of computation accuracy to allow the calculations to skip from the subsonic to supersonic region [2]

When the equations are written in hodograph form, using the velocity as independent variable, this singularity no longer occurs at $M_f = 1$, and one could hopefully integrate with complete confidence in this region Furthermore, in the subsonic portion of the nozzle, the rates of change of the gross properties with distance are quite large, and frequently a small mesh size is required The design computation time might be reduced significantly by using the hodograph form of the equations

However, the author has found that the nonequilibrium equations are unstable in many cases This note, then, is an attempt to establish a stability criterion for these equations As an illustration, the simple case of dissociating gas flow will be considered

Consider the system of equations describing the flow of an $A_2 \gtrless 2A_1$ gas in a nozzle Assume that the velocity can be used as the independent variable The equations are as follows:

Continuity

$$\frac{1}{\rho}\frac{d\rho}{du} + \frac{1}{A}\frac{dA}{du} + \frac{1}{u} = 0 \qquad (1)$$

Momentum

$$\frac{dP}{du} = -\rho u \qquad (2)$$

Energy

$$\frac{\Delta H}{M_2}\frac{d\alpha}{du} + \frac{c_P}{M_2}\frac{dT}{du} + u = 0 \qquad (3)$$

State

$$\frac{1}{P}\frac{dP}{du} = \frac{1}{\rho}\frac{d\rho}{du} + \frac{1}{T}\frac{dT}{du} + \frac{1}{1+\alpha}\frac{d\alpha}{du} \qquad (4)$$

Nozzle Shape

$$\frac{d\ln A}{du} = F(x)\frac{dx}{du} \qquad (5)$$

Kinetic Rate

$$u\frac{d\alpha}{du} = k\left\{\frac{\alpha_e^2 - \alpha^2}{\alpha^2(1+\alpha)}\right\}\frac{dx}{du} = k\Delta\frac{dx}{du} \qquad (6)$$

where α is the extent of dissociation of the dimer, α_e is the

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* Student, Department of Chemical Engineering Student Member AIAA

extent of dissociation if the reaction were in equilibrium at the local conditions, k is the dissociation reaction rate "constant," M_2 is the molecular weight of the dimer, and ΔH is the heat of dissociation of the dimer Other symbols are standard

Recognizing, from (5), that

$$\frac{dx}{du} = \frac{d\ln A}{du}\frac{1}{F(x)} \tag{7}$$

we can get an expression for the rate of reaction:

$$\frac{d\alpha}{du} = \frac{k\Delta}{uF}\frac{d\ln A}{du} \tag{8}$$

Solving (4) for $(1/\rho)(d\rho/du)$, substituting into it (2, 3, and 8), recognizing that $\gamma M_f{}^2 = \rho(u^2/P)$, $u^2M_2/TC_P = (\gamma - 1)M_f{}^2$, where γ is the ratio of specific heats, and letting $m = \Delta H/RT$, $g' = C_P/R$, and $\Gamma = \{[(1+\alpha)m]/g'\} - 1$, we obtain

$$\frac{1}{\rho}\frac{d\rho}{du} = -\frac{M_f{}^2}{u} + \frac{\Gamma}{1+\alpha}\frac{k\Delta}{uF}\frac{d\ln A}{du} \tag{9}$$

We can now eliminate $(1/\rho)(d\rho/du)$ using (1) and can arrive at an expression for†

$$\frac{d\ln A}{du} = \frac{M_f{}^2 - 1}{u}\left\{1 + \frac{\Gamma k\Delta}{(1+\alpha)uF}\right\}^{-1} \tag{10}$$

Let us define $\psi = \Gamma k\Delta/[(1+\alpha)uF]$ We can now substitute (10) into (8), obtaining a rate equation of the form

$$\frac{d\alpha}{du} = \frac{(M_f{}^2 - 1)}{u}\frac{\psi(1+\alpha)}{\Gamma(1+\psi)} \tag{11}$$

Using this equation and (3), we can get an expression for $d\ln T/d\ln u$:

$$d\ln T/d\ln u = -[(\gamma + \phi - 1)M_f{}^2 - \phi] \tag{12}$$

where $\phi = \psi(\Gamma + 1)/\Gamma(\psi + 1)$

One criterion of stability is that $d\ln T/d\ln u < 0$ Hence from (12), $(\gamma - 1 + \phi)M_f{}^2 - \phi > 0$ Solving for ϕ, ψ, and finally Δ, we get

$$\phi < [(\gamma - 1)M_f{}^2]/(1 - M_f{}^2) \tag{13}$$

$$\psi < \frac{[\Gamma/(\Gamma+1)](\gamma - 1)}{[(1 - M_f{}^2)/M_f{}^2] - [\Gamma/(\Gamma+1)](\gamma - 1)} \tag{14}$$

and

$$\Delta < \frac{(1+\alpha)uF}{k}\left\{\frac{[\Gamma/(\Gamma+1)](\gamma - 1)}{[(1 - M_f{}^2)/M_f{}^2] - [\Gamma/(\Gamma+1)](\gamma - 1)}\right\} \tag{15}$$

Note that Δ represents a measure of the deviation from equilibrium [see Eq (6)] We can rewrite (15) as follows:

$$\Delta < \frac{\mu(1+\alpha)F(x')}{D} \times \left\{\frac{[\Gamma/(\Gamma+1)](\gamma - 1)}{[(1 - M_f{}^2)/M_f{}^2] - [\Gamma/(\Gamma+1)](\gamma - 1)}\right\} \tag{16}$$

where $\mu = u/a$, $F(x') = d\ln A/dx'$, $D = lk/a$, $x' = x/l$, l is some characteristic length, and $a = P_0/\rho_0$ is Newton's velocity of sound at chamber conditions

As a numerical example, let $\mu = 0\ 04$, $M_f = 0\ 04$, $F(x') = -100$, $D = 1000$, $\alpha = 0\ 2$, $\Gamma = 4$, and $\gamma = 1\ 2$ For stability, $\phi < 4\ 01 \times 10^{-7}$, $\psi < 3\ 57 \times 10^{-3}$, and $\Delta < -4\ 29 \times 10^{-7}$

† Note that the singularity in the equations has been shifted to the point where $\psi = -1$ This point occurs in the supersonic region

It is obvious that stability of the nonequilibrium hodograph equations in the subsonic region is achieved only if the gas is extremely near equilibrium Note that these equations appear more stable the closer one gets to equilibrium, apparently behaving in a manner opposite to the physical equations [4] In general, however, the hodograph equations are not stable for most regions of interest

This stability criterion can also be applied to equilibrium and frozen flows with predicted results For equilibrium flow, it can be shown that[3]

$$\frac{d\alpha}{du} = \frac{\beta m}{T}\frac{dT}{du} - \frac{\beta}{P}\frac{dP}{du} \tag{17}$$

where $\beta = [\alpha(1 - \alpha^2)]/2$ Using this equation, along with (2) and (3), one can arrive at an equation similiar to (12):

$$\frac{d\ln T}{d\ln u} = -\frac{(\gamma^e - 1)M_e{}^2\{1 + [\beta m/(1+\alpha)]\}}{[1 + (\beta m/g')]} \tag{18}$$

where M_e is now the Mach number based on the equilibrium velocity of sound, and γ^e is defined as $\gamma^e = a\,{}^2P/\rho$, where a_e is the equilibrium velocity of sound [3] Since $\alpha, \beta, m, \gamma^e, g' > 0$ for an exothermic recombination reaction and $\gamma^e > 1$, $d\ln T/d\ln u$ is always less than zero Hence the equilibrium equations are stable everywhere

For the frozen hodograph equations, by setting $d\alpha/du = 0$, we can show that

$$d\ln T/d\ln u = -(\gamma - 1)M_f{}^2 \tag{19}$$

and again $d\ln T/d\ln u < 0$ everywhere, meaning that the frozen hodograph equations are also stable everywhere

There is little reason to believe that this stability criterion is the best, but it qualitatively agrees with the author's computing experience on these equations

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Flow in the Three-Dimensional Boundary Layer on a Spinning Body of Revolution

O Parr*

Institute of Fluid Mechanics of the Technical University, Braunschweig, Germany

THE flow on a body of revolution spinning about its axis, which is parallel to a stream, is of some practical importance The flow on a spinning projectile and on the hub of an axial turbomachine are typical examples The problem has been dealt with experimentally by Wieselsberger[1]

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* Research Collaborator